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Anomalous dissipation: Strong non-Markovian effect and its dynamical origin

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We report the effects of anomalous dissipation with a vanishing effective friction, which can induce ballistic diffusion and dissipative acceleration, where the fluctuation-dissipation theorem is fulfilled. An influence factor is introduced in order to describe the role of non-Markovian friction and the force-folded effect on the long-time results. The velocity-dependent coupling and force might be a dynamical origin of this dissipation. The steady acceleration of a particle moving in periodic and magnetic-force potentials are calculated.

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The non-Markovian friction (frequency-dependent friction) mechanism is usually described by the non-Ohmic model within the framework of the generalized Langevin equation (GLE), which arises from a spectral density with the form $J(\omega)=m\gamma_s\omega^s f(\omega/\omega_c)$, where $f(\omega/\omega_c)$ is a cutoff function [1,2]. It is concluded that the power $0<s<1$ is for subdiffusion, $1<s<2$ for superdiffusion, while for $s>2$ the mean-square displacement and the variance grow $\propto t^2$, i.e, ballistic diffusion appears [1]. Note that when $s<2$, the average velocity of a damped free particle relaxes towards to zero due to dissipation; we call this situation as a regular dissipative process. The obvious examples of $s>2$ can be found in the velocity-dependent forces for electromagnetic problems, for example, in a superconducting quantum interference device (SQUID) [3,4] and blackbody electromagnetic field [5], where the basic variable, the magnetic flux, is coupled to a quantity with the dimension of electric current. Nevertheless, the macroscopical properties of the system at long times and the validity of the Kubo fluctuation-dissipation theorem (FDT) in the ballistic process induced by anomalous dissipation with a vanishing effective friction need to be clarified.

Newton's second law in a conservative system states that the acceleration of a particle is equal to the total external force divided by the particle's mass, which, however, vanishes in usual dissipative systems, because the external force is equivalent to the friction force being proportional to the particle velocity. A problem of broad interest is whether there exists a general intermediate situation between the above-mentioned two cases. According to the Einstein relation, one has $\langle x(t)\rangle_F=F\langle x^2(t)\rangle_0/(2k_BT)$ [6], relating the first moment in the presence of force F to the second moment in the absence of the force. There should exhibit a steady acceleration if the mean-square displacement of the particle grows with the square of time. Very recently, the ballistic process has become a challenging subject, for instance, short-time ballistic displacement in a bacterial bath, ballistic aggregation, ballistic deposition, and ballistic strings [7]. In theory, ballistic diffusion (as the limit of superdiffusion) has been predicted [1,2,8–10], ballistic transport is observed in the harmonic lattice, and its effect on heat conduction was discussed in [11]. However, possible physical origins for processes of this kind have remained open.

In this Rapid Communication, an influence factor of non-Markovian friction upon long-time results is proposed and used to study the relationship between fluctuation and anomalous dissipation as well as the force-folded effect in the ballistic process. Furthermore, we want to explore what kind of heat bath could play such a role and describe such physical situations where there might exist a dissipative acceleration.

Fluctuation related to anomalous dissipation. The motion of a particle in a potential U subjected to a thermal colored noise $\varepsilon(t)$ is described by a GLE [12],

$$m\dot{v}(t)=-U'(x)-\int_0^t\beta(t-t')v(t')dt'+\varepsilon(t), \tag{1}$$

where $\beta(t)$ is the memory friction kernel, $\varepsilon(t)$ with zero mean is assumed to be uncorrelated to the initial velocity and must satisfy the second FDT [13] expressed as $\langle\varepsilon(t)\varepsilon(0)\rangle=\beta(t)\langle v^2\rangle_{\rm eq}$, in order to emphasize the ensemble to be stationary, where the subscript "eq" denotes the expectation with respect to the assumed equilibrium state.

When the potential is absent, the solution of Eq. (1) can be obtained by means of the Laplace transform technique, $v(t)=v_0\Phi(t)+1/m\int_0^t\Phi(t-t')\varepsilon(t')dt'$, where v_0 is the initial velocity of the particle. The function $\Phi(t)$ is the inverse form of the Laplace transform $\hat{\Phi}(z)=[z+\hat{\beta}(z)]^{-1}$, where $\hat{\beta}(z)$ is the Laplace transform of the memory friction kernel. We have obtained the two-time velocity correlation function in a generic form as

$$\begin{aligned}\langle v(t_1)v(t_2)\rangle &= \{v_0^2\}f_c^2+\langle v^2\rangle_{\rm eq}\Big(f_c(1-f_c)+f_c\sum_j \mathrm{res}[\hat{\Phi}(z_j)]\\&\quad\times\exp(z_j|t_1-t_2|)+\sum_{i,j}\mathrm{res}[\hat{\Phi}(z_i)]\\&\quad\times\mathrm{res}[\hat{\Phi}(z_j)]\exp(z_i|t_1-t_2|)\Big)+[\{v_0^2\}-\langle v^2\rangle_{\rm eq}]\\&\quad\times\Big(f_c\sum_j\mathrm{res}[\hat{\Phi}(z_j)][\exp(z_jt_1)+\exp(z_jt_2)]\\&\quad+\sum_{i,j}\mathrm{res}[\hat{\Phi}(z_i)]\mathrm{res}[\hat{\Phi}(z_j)]\exp(z_jt_1+z_it_2)\Big),\end{aligned} \tag{2}$$

where z_i and z_j are nonzero roots of the equation: $z+\hat{\beta}(z)=0$, $\{\,\}$ denotes the initial average and res ( ) the residue. The mean-square displacement of the free particle can be obtained to compute the double integral

$$\langle x^2(t)\rangle = \int_0^t\int_0^t \langle v(t_1)v(t_2)\rangle dt_1 dt_2. \tag{3}$$

We have the long-time asymptotic result for Eq. (3), $\langle x^2(t\to\infty)\rangle=[\langle v^2\rangle_{\rm eq}f_c+(\{v_0^2\}-\langle v^2\rangle_{\rm eq})f_c^2]t^2$. Here f_c is called the influence factor of non-Markovian friction on the result of a system at long times, which is given by $f_c=\Phi(t\to\infty)=\{1+\lim_{z\to 0}[\hat{\beta}(z)/z]\}^{-1}$. If $f_c\neq 0, \hat{\beta}(0)=\int_0^\infty\beta(t)dt=0$ should be fulfilled and leads to $\langle v(t\to\infty)\rangle=v_0 f_c\neq 0$. This implies that the effective friction strength vanishes and this is just corresponding to an *anomalous dissipative* process due to ballistic diffusion; however, $f_c=0$ for both Markovian and normal non-Markovian (subdiffusion and superdiffusions) processes.

Note that Eq. (2) consists of a v_0-dependent constant part (which does not appear in normal, subdiffusions and superdiffusions) [12], a stationary part, and an aging term (depending on the waiting time). The second moment of velocity can still reach its equilibrium value, i.e., $\langle v^2(t\to\infty)\rangle=\langle v^2\rangle_{\rm eq}+f_c^2[\{v_0^2\}-\langle v^2\rangle_{\rm eq}]$, the aging term can be removed and the velocity correlation function has a behavior of time-translation invariance at any time, as soon as the initial velocity v_0 of the particle is assumed to be in the equilibrium state. Namely, the particle velocity becomes a stationary process and then the Kubo first FDT is met. In this case one also needs not to worry about the violation of FDT, which is contrary to what is claimed in Ref. [14], where v_0 was chosen to be zero. On the other hand, when f_c is finite and v_0 is not assumed to be in the thermal equilibrium state, this is a generalized representation of a fluctuation-dissipation process, and actually Kubo originally did not consider this case in his formulation [13]. In this case the Kubo first FDT is not valid.

The velocity-dependent coupling model. In order to investigate a physical origin of both anomalous dissipation and ballistic diffusion, we present an extension for the system-plus-reservoir model [1,3,4,15,16], which includes altogether four kinds of bilinear couplings between the system and environmental degrees of freedom. The whole Hamiltonian of the system and environment reads

$$H=\frac{1}{2}m\dot{x}^2+U(x)+\sum_{j=1}^{N}\left[\frac{m_j}{2}(\dot{q}_j^2+\omega_j^2q_j^2)+g(x,\dot{x},q_j,\dot{q}_j)\right], \tag{4}$$

where x and q_j are coordinates of the system and the heat bath, respectively. For this formalism, the coupling term is $g=-c_jxq_j+c_j^2x^2/(2m_j\omega_j^2)$ for the usual coordinate coupling [1]; $g=-d_{1,j}x\dot{q}_j$ [3] or $g=-d_{2,j}\dot{x}q_j$ for the system coordinate (velocity) and environmental velocities (coordinates) coupling [3,5]; and $g=-e_j\dot{x}\dot{q}_j-e_j^2\dot{x}^2/(2m_j)$ for velocity-velocities coupling, where two additional terms appearing in the first and fourth cases are in order to compensate coupling-induced potential and mass renormalization, respectively.

Some authors [3–5,15,16] considered the velocity-coupling model to be equivalent to the coordinate-coupling one, because the velocity-dependent coupling can be transformed into a very similar Lagrangian with a coordinate coupling instead. However, we have found here that their spectra have different forms [17], when all coupling strengths are assumed to be independent of frequency, which would lead to quite different dynamical behaviors of the system at long times, because the latter is governed by the low frequency limit of the power spectrum of noise autocorrelation function. If the environmental oscillators have a s_0-power non-Ohmic spectral distribution for coordinate-coordinate coupling, the power of the spectral density for the system coordinate (velocity) and environmental velocities (coordinates) coupling should be s_0+2 and the power for velocity-velocities coupling s_0+4, so that the velocity-dependent coupling might induce ballistic diffusion.

Just for illustration and for easy simulation, we here consider a simple case that the environmental oscillators have a structure, which is a power spectrum with a narrow Lorentzian peak centered, not at zero frequency. The friction kernel $\beta_{c-c}(t)=(\eta\Omega_0^2/\Gamma)e^{-\Gamma t/2}[\cos\omega_1 t+(\Gamma/2\omega_1)\sin\omega_1 t]$ for the system coordinate and environmental coordinates coupling, where Γ and Ω_0 are the damping and frequency parameters of the harmonic noise (HN) [18]. Then for the system coordinate (velocity) and environmental velocities (coordinates) coupling, we have

$$\beta_{c-v}(t)=\eta\Gamma\,\exp(-\Gamma t/2)\left(\cos\omega_1 t-\frac{\Gamma}{2\omega_1}\sin\omega_1 t\right), \tag{5}$$

where $\omega_1^2=\Omega_0^2-\Gamma^2/4$ and η is the friction of the system corresponding to a thermal white noise. Equation (5) is the correlation function of time derivative of HN, which is called the harmonic velocity noise (HVN) here. We have $f_c=(1+\eta\Gamma\Omega_0^{-2})^{-1}$ for the coordinate (velocity)-velocities (coordinates) coupling and $f_c=(1+\eta/\Gamma)^{-1}$ for the velocity-velocities coupling; however, $f_c=0$ for the coordinate-coordinates coupling.

In reality, the value of spectrum of thermal-colored noise-induced ballistic diffusion is equal to zero at zero frequency. Moreover, any realistic spectral density of noise falls off in the limit $\omega\to\infty$, because certain physical quantities cannot diverge. Both require the noise having a *band-passing* behavior, for example, a thermal broad-band noise proposed in Refs. [10,19] as the difference between two Ornstein-Uhlenbeck noises (OUNs) with time constants τ_1 and τ_2, its $f_c=\left[1+\frac{1}{2}\eta\tau_1^2/(\tau_1+\tau_2)\right]^{-1}$; however, for thermal white noise, OUN and HN, their low-frequency parts do not vanish and then $\hat{\beta}(0)\neq 0$, so that $f_c=0$.

It is possible to reformulate Eq. (1) with (5) into a set of Markovian Langevin equations through introducing variable transformations. We numerically calculate the mean-square displacement and velocity of a free particle with a Gaussian distribution for the initial velocity with zero mean and variance $\{v_0^2\}=k_BT_0/m$, where T_0 is the initial temperature of the particle which could differ from the temperature T of heat bath. The results compared with the HN case are shown in Figs. 1(a)–1(d), here the dimensionless units ($m=1$ and $k_B=1$) are used. It is seen from Fig. 1(a) that the asymptotic results for the coordinate-coordinates coupling do not depend on the initial condition, namely, the slopes of $\langle x^2(t\to\infty)\rangle$ for various T_0 are the same and their difference is due to the term $\{v_0^2\}\eta^{-2}$, the non-Markovian effect influences only on

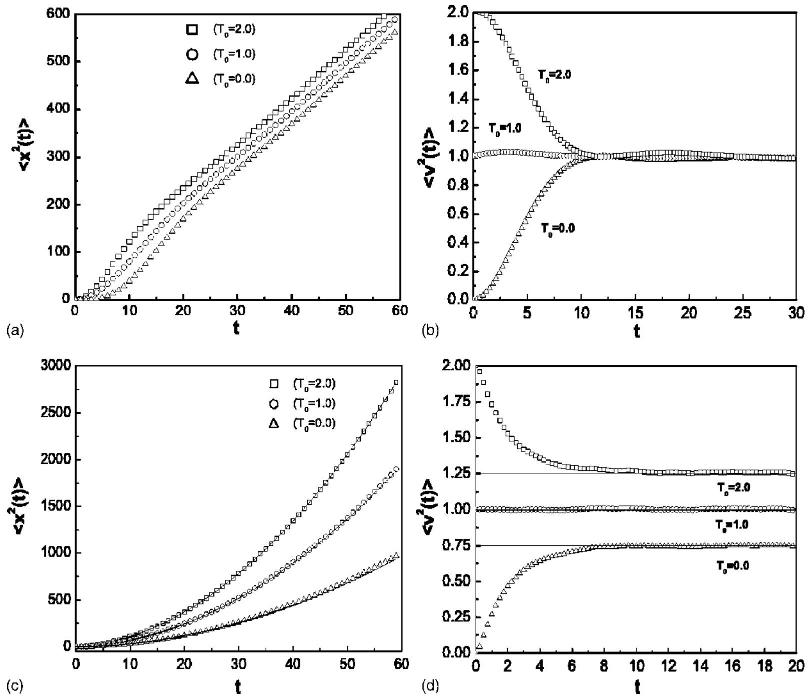

FIG. 1. Calculated time-dependent mean-square displacement and velocity of a free particle with different initial temperatures. (a) and (b) are the results of the coordinate-coordinates coupling with a Lorentzian spectrum (HN case); (c) and (d) are the results of the coordinate-velocities coupling (HVN case). The parameters used are $T=1.0,\Gamma=5.0,\Omega_0=1.0$, and $\eta=0.2$.

the transient process [see Fig. 1(b)]; while the asymptotic results for the velocity-dependent coupling are sensitive to the initial conditions [see Figs. 1(c) and 1(d)], where the numerical data are in agreement with or approach to the theoretical expressions (the solid lines) of $\langle x^2(t)\rangle$ [Equation (3) and its curvature depend on the initial velocity] and of $\langle v^2(t\rightarrow\infty)\rangle$. This allows us to classify two classes of non-Markovian processes. The first class, which we call normal one, where the velocity memory plays a role only in the transient process, and the second class, which we call the *strong non-Markovian process*, where the velocity memory, depending on the initial condition, influences the long time results.

The dissipative acceleration. If we add an external force F into Eq. (1), only a steady velocity appears for normal Brownian motion. However, in the present model of ballistic diffusion, the acceleration of the particle automatically appears as

$$a_c = \frac{F}{m} f_c. \tag{6}$$

The proposed factor f_c with $0 \leqslant f_c \leqslant 1$, in fact, can be expressed as the additional mass term $\Delta m = m_r - m = m(1-f_c)/f_c$ in the mass renormalization scheme [1] where the effect of the heat bath on the dynamics at long times is to renormalize the particle's mass, or, described as a force-folded effect in the Newton's equation if the particle's mass is assumed to be a constant in a dissipative environment.

The acceleration of the particle moving in a titled periodic potential [$U(x)=-U_0 \sin x - Fx$] [20] is shown in Fig. 2, which is a function of the constant force F. When $F/U_0 > 1$, the titled potential has no local minimum and the particle starts to get an acceleration along the direction of external force at low temperatures. When the temperature increases, the thermal fluctuations help the particle to climb over energy barriers, since there still exist fluctuations without effective friction on the average. Thus the particle can have a directional acceleration even for a small force. However, there exists only a steady velocity along the direction of external force in normal dissipative systems [20].

A real physical example for $s=3$ is the vortex diffusion, where the magnetic force depends on the relative velocity between the superfluid velocity and the vortex velocity, and will contribute to the vortex potential [21]. We write down the classical equation of motion for a charged particle with mass m and charge q as [3,21]

$$m\ddot{\mathbf{R}} = -\nabla U(\mathbf{R}) - \eta\dot{\mathbf{R}} + \frac{q}{c}\dot{\mathbf{R}} \times \mathbf{B} + \xi(t), \tag{7}$$

where ξ is a zero-mean Gaussian white noise. The scalar potential U for the charged particle has the form $U = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2) - Fz$ and the external magnetic field is tilted $\mathbf{B}=(B,0,0)$. After eliminating the x and y degrees of

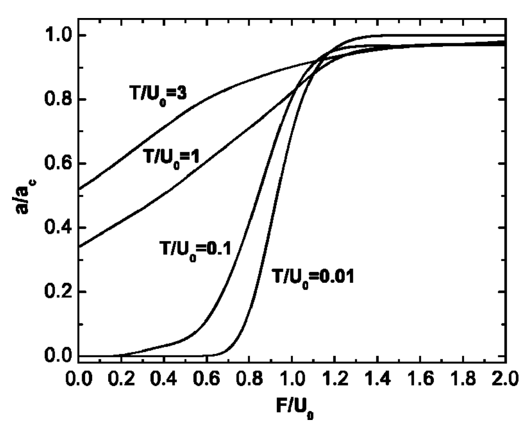

FIG. 2. The acceleration as a function of F/U_0 for various temperatures T/U_0. The parameters used are $\Gamma=3.0,\Omega_0=1.0$, and $\eta=2.0$.

freedom, the problem can be mapped onto an effective one-dimensional GLE when normal friction is absent ($\eta=0$ and without thermal noise), i.e.,

$$m\ddot{z}(t) + \frac{B^2}{m}\int_0^t \cos\,\omega_y(t-t')\dot{z}(t')dt' = \varepsilon(t) + F \qquad (8)$$

with $\varepsilon(t)=B[y(0)\omega_y\sin\,\omega_y t-\dot{y}(0)\cos\,\omega_y t]$. Assuming that $\{y(0)\}=\{\dot{y}(0)\}=0,\{\dot{y}^2(0)\}=\omega_y^2\{y^2(0)\}=k_BT/m$, and $\{y(0)\dot{y}(0)\}=0$, the anomalous memory friction kernel function $\beta(t)$ is connected to the correlation of the noise $\varepsilon(t)$ by the second FDT as the initial distribution of $\varepsilon(t)$ is assumed to be a Gaussian function with $\langle\varepsilon(t)\varepsilon(t')\rangle=(B^2/m)k_BT\cos\,\omega_y(t-t')$. We have

$$\dot{\Phi}(t) = \frac{(m\omega_y)^2}{B^2+(m\omega_y)^2} + \frac{B^2}{B^2+(m\omega_y)^2}\cos\Big[\sqrt{\omega_y^2+(B/m)^2}t\Big], \qquad (9)$$

and the time-average acceleration of the charged particle is determined by

$$\overline{a_c} = \lim_{t\to\infty}\frac{1}{\Lambda}\int_t^{t+\Lambda} dt'\frac{F}{m}\dot{\Phi}(t') = \frac{F}{m}\frac{(m\omega_y)^2}{B^2+(m\omega_y)^2}, \qquad (10)$$

where Λ is the time period of Eq. (9). Here the nontransport degrees of freedom act as the role of the heat bath with a finite monochromatic spectrum $\delta(\omega-\omega_y)$. The correction factor of the average acceleration is given by $f_c=\{1+[B/(m\omega_y)]^2\}^{-1}$ and thus $\overline{a_c}\to 0$ if $B\to\infty$ even in the absence of dissipation. This is a phenomenon of magnetics-induced damping [21].

In summary, we have found that velocity-dependent coupling and corresponding to a thermal band-passing noise can induce ballistic diffusion, where the effective friction strength of the system vanishes. Both the transient process and the long-time result for a system of this kind are sensitive to the initial conditions. We have carefully examined the validity of the Kubo first FDT by assuming that the Kubo second FDT is fulfilled. It is shown that the former is also valid for the ballistic diffusion process as if the initial particle velocity is taken to be the equilibrium distribution. The physical situation can be found in the vortex transport in the presence of magnetic field. Indeed, the mean velocity of a damped free particle should not relax towards to zero, and when an external constant force is added into the system, it will be equilibrated partly by anomalous memory friction and lead to a force-folded effect. Thus the steady acceleration appears in such anomalous dissipative systems. This can be used as a probe to investigate characteristic behaviors of the environment and interaction form through calculating asymptotic results of the system.

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